On Solving Regular Linear Rational Expectations Models

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Abstract

We present a computationally effective method of solving regular linear dynamic systems based on Schur decomposition. We also derive first derivative of matrices describing model dynamics with respect to model parameters.

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1 Introduction

We present a method of solving regular linear rational expectations models in general form based on generalized Schur decomposition. Such a models arises naturally, using the perturbation technique of solving dynamic economies.

Presented method is rather standard. See e.g. [2]. There are a few packages implementing this technique. This methods however differs substantially from existing solutions techniques. The most important differences is lack of more detailed structure of the models, we do not analyze predefined state variables explicitly. In many cases determining state variables is complicated, especially in case of large models. Additionally, existing techniques generally breaks down is the set of predefined state variables is lower, than dimension of states in the model or when predefined state variables cannot take arbitrary values in given period. The first case arises naturally in models with indeterminacy. The second problem also arise often in large models and is usually hard to detect. Analyzing unstructurized models avoids this difficulties. We present however methods of reformulating model dynamics in terms of predefined states.

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Analyzing structurized models allows for more efficient solutions in case of analyzing explicitly exogenous state variables. However computational advantage of analyzing such models in not high and introduces many difficulties. Solving models with predefined exogenous states requires solving generalized Sylvester equations. Current implementations of methods of solving such problems work only if there is exactly one solution to this equation. However generally this is not the case.

Presented technique delivers all solutions to the model or shows that there is no solution.

In order to allow analyzing large dynamic models we present a technique of reduction the problem dimension in case of very large number of endogenous variables but small or moderate number of state variables.

In many cases solving rational expectations models is not the last step and they require parameter estimation, which is extremely costly. In order to reduce this cost we present how to calculate differentials of solution with respect to model parameters.

This paper is organized as follows: section 2 presents the problem, section 3 analyzes a matrix equation determining solution to deterministic part of the models, section 4 analyzes stochastic part of the model, in section 5 we presents the method of reducing problem dimension, section 6 delivers differentials of solution with respect to additional parameters, in section 7 we present a method of expressing model dynamics in terms of predefined set of state variables.

2 The Problem

Let us consider the following linear system

$$\begin{aligned} 0 = {} & A_1 y_t + A_2 y_{t+1} + A_3 E_t y_{t+1} + B_1 z_t + B_2 z_{t+1} + B_3 E_t z_{t+1} \\ & + V_1 \epsilon_t + V_2 \epsilon_{t+1} \end{aligned} \tag{1}$$

where y is a vector of control variables, ϵ_t is a vector of i.i.d. random variables with zero mean, z_t is a vector of deterministic or stochastic exogenous shocks and is taken as given. Each variable with time subscript t belongs to information set in period t. Information set in period t contains also state variables in period t, u_t and sunspot shocks ω_t. We will describe these variables later.

Exogenous shocks do not allow for closed for solution, since solution depends on all future realizations of shocks. However models with exogenous shocks can be treated as a preprocessing step in solving large scale models with many agents. Then model in the form (1) represents first order conditions of agents of given type, which depends on additional variables. These variables are predetermined from perspective of the agent.

We assume that matrices A_1, A_2, A_3 are square and the matrix pair $(A_1, A_2 + A_3)$ is regular.

Definition 2.1. *A matrix pair* (A, B) *is regular if there exist scalars* $\alpha, \beta \in \mathcal{C}$ *such that* $\det(\alpha A - \beta B) \neq 0$.

Remark 2.2. *Regular models are the largest set of linear rational expectations models that can be solved using the generalized Schur decomposition. If a matrix pair* (A, B) *is not regular then generalized Schur decomposition is not reliable. Small perturbation of matrices* A, B *may drastically change eigenvalues of* (A, B). *Consider for example the matrix pair* $(0, 0)$. *Then perturbed matrix pair* (ϵ_1, ϵ_2) *may have arbitrary eigenvalue* $\lambda = \epsilon_1/\epsilon_2$.

We are looking for a solution in the form

$$\begin{aligned} u_{t+1} &= Pu_t + Q_1\epsilon_t + Q_2\epsilon_{t+1} + Q_3\omega_t + Q_4\omega_{t+1} + \sum_{k=0}^{\infty} F_1^k E_t z_{t+k} \\ &\quad + \sum_{k=0}^{\infty} F_2^k (E_{t+1} z_{t+1+k} - E_t z_{t+1+k}) \qquad (2) \\ y_t &= Ru_t + S_1\epsilon_t + S_3\omega_t + \sum_{k=0}^{\infty} G_1^k E_t z_{t+k} \end{aligned}$$

where $u \in \mathbb{R}^k$ is vector of state variables and $\omega_t \in \mathbb{R}^r$ is any stochastic process, such that

$$E_{t+1+k}\omega_{t+k} = 0$$

for any $k \geq 0$, possibly dependent on ϵ_t. We require that endogenous variables y_t depend only on variables, that belong to information set in period t and state variables in period $t+1$ depend only on variables belonging to information set in period $t+1$. There exist additional solutions, if we consider solutions in less restrictive form, e.g. a very reach set of nonlinear solutions.

Additionally we assume the following growth restriction:

Assumption 2.3. *We are looking for linear solutions to the system (1) such that the following growth restriction holds*

$$\lim_{t\to\infty} E_0 \left\{ \xi^t y_t \right\} = 0 \qquad (3)$$

for any u_0.

Substituting (2) to (1) yields

$$
\begin{aligned}
0 = &\Big(A_1 + (A_2 + A_3)RP\Big)u_t + \Big(V_1 + A_1S_1 + (A_2 + A_3)RQ_1\Big)\epsilon_t \\
&+ (V_2 + A_2S_1 + A_2RQ_2)\epsilon_{t+1} \\
&+ \Big(A_1S_3 + (A_2 + A_3)RQ_3\Big)\omega_t + \Big(A_2S_3 + A_2RQ_4\Big)\omega_{t+1} \\
&+ \Big(A_1G_1^0 + (A_2 + A_3)RF_1^0 + B_1\Big)z_t + (A_2RF_2^0 + A_2G_1^0 + B_2)(z_{t+1} - E_tz_{t+1}) \\
&+ \Big(A_1G_1^1 + (A_2 + A_3)RF_1^1 + (A_2 + A_3)G_1^0 + (B_2 + B_3)\Big)E_tz_{t+1} \\
&+ \sum_{k=0}^{\infty}\Big(A_1G_1^{k+2} + (A_2 + A_3)RF_1^{k+2} + (A_2 + A_3)G_1^{k+1}\Big)E_tz_{t+2+k} \\
&+ \sum_{k=0}^{\infty}(A_2RF_2^{k+1} + A_2G_1^{k+1})(E_{t+1}z_{t+2+k} - E_tz_{t+2+k})
\end{aligned}
\tag{4}
$$

equation (4) must be fulfilled for all u_t

$$0 = A_1 + (A_2 + A_3)RP \tag{5}$$

equation (4) must also be fulfilled for all ϵ_t and ω_t, hence

$$0 = V_1 + A_1S_1 + (A_2 + A_3)RQ_1 \qquad 0 = A_1S_3 + (A_2 + A_3)RQ_3 \tag{6}$$

$$0 = V_2 + A_2S_1 + A_2RQ_2 \qquad 0 = A_2S_3 + A_2RQ_4 \tag{7}$$

finally equation (4) must also be fulfilled for all E_tz_{t+k} and $E_{t+1}z_{t+1+k} - E_tz_{t+1+k}$ for all k, hence

$$
\begin{aligned}
0 &= A_1G_1^0 + (A_2 + A_3)RF_1^0 + B_1 \\
0 &= A_2RF_2^0 + A_2G_1^0 + B_2 \\
0 &= A_1G_1^1 + (A_2 + A_3)RF_1^1 + (A_2 + A_3)G_1^0 + (B_2 + B_3) \\
0 &= A_1G_1^{k+2} + (A_2 + A_3)RF_1^{k+2} + (A_2 + A_3)G_1^{k+1} \\
0 &= A_2RF_2^{k+1} + A_2G_1^{k+1}
\end{aligned}
\tag{8}
$$

The transversality condition implies that

$$\lim_{t\to\infty} \xi^t(RE_0u_t + \tilde{z}_t^2) = 0$$

where

$$\tilde{z}_t^1 = E_0\sum_{k=0}^{\infty} F_1^kE_tz_{t+k} \qquad \tilde{z}_t^2 = E_0\sum_{k=0}^{\infty} G_1^kE_tz_{t+k}$$

From (2) we have

$$E_0 u_t = P^t u_0 + \sum_{k=1}^{t} P^{k-1} \tilde{z}^1_{t-k}$$

hence

$$\lim_{t\to\infty} \xi^t (RP^t u_0 + \sum_{k=1}^{t} P^{k-1} \tilde{z}^1_{t-k} + \tilde{z}^2_t) = 0 \tag{9}$$

Equation (9) must be fulfilled for all u_0 and $\tilde{z}^1_t$, $\tilde{z}^2_t$ are independent on u_0, hence the necessary condition for holding (9) is

$$\lim_{t\to\infty} \xi^t RP^t = 0 \tag{10}$$

3 The matrix equation $\mathcal{A}R = \mathcal{B}RP$

In this section we briefly present a method of solving the matrix equation $\mathcal{A}R = \mathcal{B}RP$ for regular matrix pair (A, B).

In this section we are looking for matrices R and P, such that for a given square matrices $\mathcal{A}$ and $\mathcal{B}$ the equation $\mathcal{A}U = \mathcal{B}U\Sigma$ holds and the transversality condition $\lim_{t\to\infty} \xi^t RP^t = 0$ is satisfied.

Theorem 3.1. ***The generalized Schur decomposition.*** *For each matrices $\mathcal{A}, \mathcal{B} \in R^{n\times n}$ there exist orthogonal matrices U, V, and real matrices R_A, R_B, such that R_B is upper-triangular, R_A is quasi-upper triangular and*

$$\mathcal{A}U = VR_A \qquad\qquad \mathcal{B}U = VR_B$$

Additionally, eigenvalues of R_A, R_B can be sorted in any order.

Proof. See [1]. □

Let us assume that a matrix pair $(\mathcal{A}, \mathcal{B})$ is regular[1]. Let us consider generalized Schur decomposition of the matrix pair $(\mathcal{A}, \mathcal{B})$

$$V'\mathcal{A}U = T_A \qquad\qquad V'\mathcal{B}U = T_B$$

where matrices U and V are orthogonal, the matrix T_A is quasi-upper triangular, and the matrix T_B is upper triangular. Such a decomposition always exists. Let λ^A_i, λ^B_i are i-th eigenvalues of T_A and T_B respectively.

[1] This assumption guarantees that the Schur decomposition is numerically stable. In opposite case a matrix pair $(\mathcal{A}, \mathcal{B})$ has infinitely many eigenvalues. Small perturbations of $(\mathcal{A}, \mathcal{B})$ may drastically change matrices T_A and T_B.

Proposition 3.2. *If the matrix pair* $(\mathcal{A}, \mathcal{B})$ *is regular and* $\lambda_i^B = 0$*, then* $\lambda_i^A \neq 0$.

Proof. It results directly from the definition of regular matrix pair and from the observation, that for any $\alpha, \beta \in \mathcal{C}$, $\det(\alpha\mathcal{A} - \beta\mathcal{B}) = \det(\alpha T_A - \beta T_B)$. □

Let $\lambda_i = \lambda_i^A / \lambda_i^B$ and let λ is a set of all distinct finite eigenvalues λ_i. Let q is a size of the set λ.

Let us sort eigenvalues of T_A and T_B is such a way that all eigenvalues λ_i, such that $|\xi\lambda_i| < 1$ appears in left upper block of T_A and T_B. Then

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_A & T_{12}^A \\ 0 & T_{22}^A \end{bmatrix} = \mathcal{A} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$
$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_B & T_{12}^B \\ 0 & T_{22}^B \end{bmatrix} = \mathcal{B} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where R_A is quasi-upper triangular, R_B is upper-triangular, both matrices have the same size. This implies

$$\mathcal{A}U_1 = V_1 R_A \qquad \mathcal{B}U_1 = V_1 R_B \tag{11}$$

By assumption, the matrix R_B is invertible. Thus,

$$\mathcal{A}U_1 = \mathcal{B}U_1 (R_B)^{-1} R_A$$

Then we can take $P = (R_B)^{-1} R_A$ and $R = U_1$. If matrices R, P satisfy the equation $\mathcal{A}R = \mathcal{B}RP$, then matrices $\tilde{R} = R\Xi$, $\tilde{P} = \Xi^{-1} P \Xi$ also satisfy this equation for any invertible matrix Ξ.

Proposition 3.3. $\ker BR = 0$

Proof. Let $x \in \ker B$. Then $0 = BRx = V_1 R_B x$. V_1 has full column rank, hence $R_B x = 0$. The matrix R_B is invertible, thus $x = 0$. □

Proposition 3.4. T_{22}^A *is an invertible matrix.*

Proof. If for some i, $\lambda_i^A = 0$. Then $\lambda_i^B \neq 0$, and $\lambda_i = 0$. Hence, all zero eigenvalues of the matrix $\mathcal{A}$ are selected, and T_{22}^A contains only nonzero eigenvalues. □

Proposition 3.5. *The transversality condition* $\lim_{t\to\infty} \xi^t R P^t = 0$ *holds. If additional eigenvalue is selected in constructing matrices* P *and* R*, then* $\lim_{t\to\infty} \xi^t R P^t \neq 0$ *if exists.*

Proof. By construction all eigenvalues of the matrix ξP are lower in absolute value than unity. Thus $\lim_{t\to\infty} (\xi P)^t = 0$. If additional eigenvalue is selected in constructing the matrix P and R then one of eigenvalues of ξP is greater or equal unity. Then $\lim_{t\to\infty} (\xi P)^t \neq 0$ if the limit exists. By construction $\ker R = 0$, hence also $\lim_{t\to\infty} R(\xi P)^t \neq 0$. □

Proposition 3.6. *If all zero generalized eigenvalues are selected in constructing the matrix R in the section 3 and the matrix pair $(A_1, A_2 + A_3)$ is regular, then*

1. *the matrix* $\begin{bmatrix} A_1 & (A_2 + A_3)R \end{bmatrix}$ *has full row rank*

2. $\mathrm{null}(\begin{bmatrix} A_1 & (A_2 + A_3)R \end{bmatrix}) = \mathrm{col}(R, P)$

Proof. If all zero generalized eigenvalues are selected in constructing the matrix R, then the matrix T_{22}^A is an invertible matrix.

The matrix $\begin{bmatrix} A_1 & (A_2 + A_3)R \end{bmatrix}$ has full row rank if and only if the matrix $C = V' \begin{bmatrix} A_1 & (A_2 + A_3)R \end{bmatrix}$ has full row rank. But

$$C = \begin{bmatrix} R_A U_1' + T_{12}^A U_2' & \tilde{B}_1 \\ T_{22}^A U_2' & 0 \end{bmatrix}$$

where matrices $\tilde{B}_1$, $T_{22}^A U_2'$ are square and have full row rank. Thus, also the matrix C has full row rank. The matrix $\tilde{B}_1$ is invertible, since the matrix pair (A_1, A_2+A_3) is regular.

From the construction of matrices R and P, $A_1 R + (A_2 + B_2)RP = 0$. Since the matrix $[A_1, (A_2 + A_3)R]$ has full row rank, thus $\dim_2 \mathrm{null}([A_1, (A_2 + A_3)R]) = \dim_2 R$. Hence, $\mathrm{col}(R, P)$ spans the null space of $[A_1, (A_2 + A_3)R]$. □

4 The stochastic part

Let us consider a matrix equation $AX + B = 0$ for any matrices A, B. We are going to find all solutions to this equation. Let U is such an invertible matrix[2] that

$$UA = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$

where A_1 is a matrix with full rank with q_1 rows. Let $UB = \mathrm{col}(B_1, B_2)$ be the corresponding partition of the matrix UB. Then

$$\begin{bmatrix} A_1 \\ 0 \end{bmatrix} X = - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

If the matrix B_2 is not a zero matrix, then there is no solution to the equation $AX + B = 0$. Assume that this is not the case. We have

$$A_1 X + B_1 = 0;$$

[2]To obtain efficiently the matrix U one can consider qr or lu decomposition of the matrix A. One can also apply svd decomposition, which deliver more accurate results, but is much slower.

If A_1 is a square matrix then A_1 is invertible and $X = A_1^{-1}B_1$ is the only solution. Let us assume that A_1 is not square and V is such an invertible matrix that

$$A_1 V = \left[\begin{array}{cc} A_2 & 0 \end{array} \right]$$

where A_2 is a square matrix with q_1 rows. Then

$$\left[\begin{array}{cc} A_2 & 0 \end{array} \right] V^{-1}X + B_1 = 0;$$

Let $V^{-1}X = \mathrm{col}(X_1, X_2)$ be a partition of the matrix $V^{-1}X$ such that $\dim_1 X_1 = q_1$. Then

$$A_2 X_1 + B_1 = 0$$

and X_2 is any matrix. The matrix A_2 is invertible thus

$$X_1 = -A_2^{-1}B_1$$

and

$$X = V \left[\begin{array}{c} -A_2^{-1}B_1 \\ X_2 \end{array} \right] = -V_1 A_2^{-1} B_1 + V_2 X_2$$

where X_2 is any matrix with appropriate size, and $V = [V_1, V_2]$.

Equations (6) can be restated as

$$0 = V_1 + \left[\begin{array}{cc} A_1 & (A_2 + A_3)R \end{array} \right] \left[\begin{array}{c} S_1 \\ Q_1 \end{array} \right] \quad 0 = \left[\begin{array}{cc} A_1 & (A_2 + A_3)R \end{array} \right] \left[\begin{array}{c} S_3 \\ Q_3 \end{array} \right] \tag{12}$$

From the proposition 3.6 we have, that equations under (12) always have at least one solution if all zero eigenvalues are selected in constructing the matrix R. Thus, there exists matrices Ξ_1, Ξ_2, Ψ_1, Ψ_2 such that

$$S_1 = \Xi_1 V_1 + \Psi_1 Y_1 \qquad S_3 = \Psi_1 Y_2 \tag{13}$$

$$Q_1 = \Xi_2 V_1 + \Psi_2 Y_1 \qquad Q_3 = \Psi_2 Y_2 \tag{14}$$

are all solutions to (12), where Y_1, Y_2 are any matrices with appropriate size. Then equation (7) takes the form

$$0 = A_2 \left[\begin{array}{cc} R & \Psi_1 \end{array} \right] \left[\begin{array}{c} Q_2 \\ Y_1 \end{array} \right] + A_2 \Xi_1 V_1 + V_2 \tag{15}$$

$$0 = A_2 \left[\begin{array}{cc} R & \Psi_1 \end{array} \right] \left[\begin{array}{c} Q_4 \\ Y_2 \end{array} \right] \tag{16}$$

In general case not for any values of Y_1, Y_2 equations (7) are fulfilled. However Y_1, Y_2 can take any values if all zero eigenvalues are selected in constructing the matrix R.

Proposition 4.1. *If all zero generalized eigenvalues are selected in constructing the matrix R in the section 3, then if there exists a solution to (15-16), then there exist solutions to (15-16) for any values of Y_1, Y_2.*

Proof. Observe that $\mathrm{col}(\Psi^1, \Psi^2) = \mathrm{null}([A_1, (A_2 + A_3)R])$. From the proposition 3.6 we have $\Psi^1 = R$. Now, let $\tilde{Q}_2$, $\tilde{Y}_1$, $\tilde{Q}_4$, $\tilde{Y}_2$ solves (15-16) and let $\bar{Y}_1$, $\bar{Y}_2$ are any matrices. Then

$$\begin{aligned}0 &= A_2 R Q_2 + A_2 \Psi_1 \bar{Y}_1 + A_2 \Psi_1 (\tilde{Y}_1 - \bar{Y}_1) + A_2 \tilde{S}_1 + V_2 \\ &= A_2 R (Q_2 + (\tilde{Y}_1 - \bar{Y}_1)) + A_2 \Psi_1 \bar{Y}_1 + A_2 \tilde{S}_1 + V_2 \\ 0 &= A_2 R Q_4 + A_2 \Psi_1 \bar{Y}_2 + A_2 \Psi_1 (\tilde{Y}_2 - \bar{Y}_2) = A_2 R (Q_4 + (\tilde{Y}_2 - \bar{Y}_2)) + A_2 \Psi_1 \bar{Y}_2\end{aligned}$$

Then $\bar{Q}_2$, $\bar{Y}_1$, $\bar{Q}_4$, $\bar{Y}_2$ solves (15-16), where $\bar{Q}_2 = Q_2 + (\tilde{Y}_1 - \bar{Y}_1)$, $\bar{Q}_4 = Q_4 + (\tilde{Y}_2 - \bar{Y}_2)$. □

By the proposition (4.1) we can treat matrices Y_1, Y_2 as given in equations (15-16), and solve these equations with respect to Q_2 and Q_4 only.

If $\dim_2 R > 0$, then $\Psi_1, \Psi_2 \neq 0$ and there are infinitely many possible choices for S_1 and S_3. However this indeterminacy is not the true indeterminacy, but results from this, that we have not imposed any structure on state variables u_t.

Proposition 4.2. *For any values of Y_1, Y_2 we can choose a new representation of solution, which implies the same dynamics of endogenous variables and in which $Y_1, Y_2 = 0$.*

Proof. Endogenous variables, y_t satisfies

$$\begin{aligned}y_t &= R u_t + \Xi_1 V_1 \epsilon_t + \Psi_1 Y_1 \epsilon_t + \Psi_1 Y_2 \omega_t + \sum_{k=0}^{\infty} G_1^k E_t z_{t+k} \\ &= R u_t + \Xi_1 V_1 \epsilon_t + R Y_1 \epsilon_t + R Y_2 \omega_t + \sum_{k=0}^{\infty} G_1^k E_t z_{t+k} \\ &= R(u_t + Y_1 \epsilon_t + Y_2 \omega_t) + \Xi_1 V_1 \epsilon_t + \sum_{k=0}^{\infty} G_1^k E_t z_{t+k} \equiv R\tilde{u} + \Xi_1 V_1 \epsilon_t + \sum_{k=0}^{\infty} G_1^k E_t z_{t+k}\end{aligned}$$

where $\tilde{u} = u_t + Y_1\epsilon_t + Y_2\omega_t$. Then, from (2)

$$\begin{aligned}\tilde{u}_{t+1} &= P\tilde{u}_t + (Q_1 - PY_1)\epsilon_t + (Q_2 + Y_1)\epsilon_{t+1} + (Q_3 - PY_2)\omega_t + (Q_4 + Y_2)\omega_{t+1} \\ &\quad + \sum_{k=0}^{\infty} F_1^k E_t z_{t+k} + \sum_{k=0}^{\infty} F_2^k (E_{t+1} z_{t+1+k} - E_t z_{t+1+k}) \\ &= P\tilde{u}_t + \Xi_2 V_1 \epsilon_t + (Q_2 + Y_1)\epsilon_{t+1} + (Q_4 + Y_2)\omega_{t+1} + \sum_{k=0}^{\infty} F_1^k E_t z_{t+k} \\ &\quad + \sum_{k=0}^{\infty} F_2^k (E_{t+1} z_{t+1+k} - E_t z_{t+1+k})\end{aligned}$$

where we have used that $\Psi_2 = P$. From the proof of proposition 4.1 we have that $Q_2 + Y_1 = \bar{Q}_2$, $Q_4 + Y_2 = \bar{Q}_4$, where $\bar{Q}_2$, $\bar{Q}_4$ solve equations (15-16) for $Y_1, Y_2 = 0$. Hence, this representation implies the same dynamics of endogenous variables as the original representation and can be found by assuming that $Y_1, Y_2 = 0$. □

Further we assume that $Y_1, Y_2 = 0$, hence $Q_3, S_3 = 0$. We have

$$0 = A_2 R Q_2 + A_2 \Xi_1 V_1 + V_2 \qquad\qquad 0 = A_2 R Q_4 \tag{17}$$

Applying again the procedure described earlier we obtain that if solutions to (17) exist then there exist matrices Ξ_3 and Ψ_3 such that all solutions to (17) take the form

$$Q_2 = \Xi_3(A_2 \Xi_1 V_1 + V_2) + \Psi_3 Z_1 \qquad\qquad Q_4 = \Psi_3 Z_2$$

Since we have allowed for the sunspot shock, ω_t to be correlated with ϵ_t, we can assume that $Z_1 = 0$. We can also assume that Z_2 is an identity matrix. In the opposite case we can take $\tilde{\omega}_t = Z_2 \omega_t$ as a new sunspot shock. In this way we obtain all solutions to (6-7).

Remark 4.3. *In general case the equation (17) need not have any solution. Consider for example a model*

$$0 = -2x_t + x_{t+1} + \epsilon_t \tag{18}$$

This model is regular but does not have any stable solutions, hence $\dim_2 u_t = 0$ *and equation (12) takes the form* $0 = 1 - 2S_1$*, hence* $S_1 = 1/2$*. Now, the equation (17) is equivalent to* $0 = 1/2$*, and there are no solutions to (18).*

5 The exogenous part

Equations (8) can be restated as

$$
\begin{aligned}
0 &= \left[\begin{array}{cc} A_1 & (A_2+A_3)R \end{array}\right] \left[\begin{array}{c} G_1^0 \\ F_1^0 \end{array}\right] + B_1 \\
0 &= \left[\begin{array}{cc} A_2 & A_2 R \end{array}\right] \left[\begin{array}{c} G_1^0 \\ F_2^0 \end{array}\right] + B_2
\end{aligned} \tag{19}
$$

The first equation under (19) always have at least one solution if all zero eigenvalues are selected in constructing the matrix R, thus G_1^0, F_1^0 take the form

$$
G_1^0 = \Xi_1 B_1 + \Psi_1 X_1^1 \qquad\qquad F_1^0 = \Xi_2 B_1 + \Psi_2 X_2^1
$$

Similarly as in case of stochastic part, matrices X_1^1, X_2^1 can take any value. Additionally for given values of X_1^1, X_2^1 we can choose a new representation of state dynamic, which implies the same dynamics of endogenous variables y_t and this representation can be found by assuming that $X_1^1, X_2^1 = 0$. Hence, without loss of generality we can assume that $X_1^1 = 0$, $X_2^1 = 0$.

The second equation under (19) takes now the form

$$
0 = A_2 R F_2^0 + A_2 \Xi_1 B_1 + B_2
$$

and if this equation has any solution then this solution takes the form

$$
F_2^0 = \Xi_3 (A_2 \Xi_1 B_1 + B_2) + \Psi_3 X_2^2
$$

The indeterminacy introduced by this equation can be included in the sunspot shock ω_{t+1} by taking $\tilde{\omega}_{t+1} = \omega_{t+1} + X_2^2(z_{t+1} - E_t z_{t+1})$ as a new sunspot shock. Hence, we can assume that $X_2^2 = 0$. We do not loose in this way any additional source of indeterminacy, since we did not impose any restriction on the process ω_t.

Further, from (8) we have

$$
\begin{aligned}
0 &= \left[\begin{array}{cc} A_1 & (A_2+A_3)R \end{array}\right] \left[\begin{array}{c} G_1^1 \\ F_1^1 \end{array}\right] + (A_2+A_3)G_1^0 + (B_2+B_3) \\
0 &= \left[\begin{array}{cc} A_2 & A_2 R \end{array}\right] \left[\begin{array}{c} G_1^1 \\ F_2^1 \end{array}\right]
\end{aligned} \tag{20}
$$

Matrices G_1^1, F_1^1 takes the form

$$
\begin{aligned}
G_1^1 &= \Xi_1((A_2+A_3)G_1^0 + (B_2+B_3)) + \Psi_1 X_1^3 \\
F_1^1 &= \Xi_2((A_2+A_3)G_1^0 + (B_2+B_3)) + \Psi_2 X_2^3
\end{aligned}
$$

again assuming that all zero eigenvalues have been selected in constructing the matrix R, then there exist solutions to (20) and we can take $X_1^3 = 0$, $X_2^3 = 0$ without loss of generality. Then the second equation under (20) takes the form

$$0 = A_2 R F_2^1 + A_2 \Xi_1((A_2 + A_3)G_1^0 + (B_2 + B_3))$$

which has solution (if exists)

$$F_2^1 = \Xi_3 A_2 \Xi_1((A_2 + A_3)G_1^0 + (B_2 + B_3)) + \Psi_3 X_2^4$$

again we assume that $X_2^4 = 0$ by taking $\tilde{\omega}_{t+1} = \omega_{t+1} + X_2^4(E_{t+1}z_{t+2} - E_t z_{t+2})$ as a new sunspot shock.

Finally, from (8) we have

$$\begin{aligned} 0 &= \begin{bmatrix} A_1 & (A_2 + A_3)R \end{bmatrix} \begin{bmatrix} G_1^{k+2} \\ F_1^{k+2} \end{bmatrix} + (A_2 + A_3)G_1^{k+1} \\ 0 &= \begin{bmatrix} A_2 & A_2 R \end{bmatrix} \begin{bmatrix} G_1^{k+2} \\ F_2^{k+2} \end{bmatrix} \end{aligned} \tag{21}$$

for $k \geq 0$. Then

$$\begin{aligned} G_1^{k+2} &= \Xi_1(A_2 + A_3)G_1^{k+1} + \Psi_1 X_1^5 \\ F_1^{k+2} &= \Xi_2(A_2 + A_3)G_1^{k+1} + \Psi_2 X_2^5 \end{aligned}$$

We can assume that $X_1^5 = 0$, $X_2^5 = 0$. Then

$$F_2^{k+2} = \Xi_3 A_2 \Xi_1 (A_2 + A_3) G_1^{k+1} + \Psi_3 X_2^6$$

where we can assume that $X_2^6 = 0$. Observe that if

$$A_2' \,\mathrm{null}(R' A_2') = 0$$

then there exist solutions to (20) and (21) for any $k \geq 0$. This condition is not however necessary and need not to be fulfilled in general case.

Remark 5.1. *Solution to (1) described by matrices G_1^k, F_1^k, F_2^k need not satisfy the growth restriction condition (3). Stability of the solution depends on properties of the process z_t.*

6 Problem reduction

Usually the matrix $A_2 + A_3$ has large null space especially in case of large models. We can use this property to decrease computation cost of solving the problem (5). Let Q is an orthogonal matrix such that

$$Q'(A_2 + A_3) = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \equiv \tilde{B}$$

and let $Q'A_1 = \mathrm{col}(\tilde{A}_1^1, \tilde{A}_1^2) \equiv \tilde{A}$ be corresponding partition of the matrix $Q'A_1$. Then we have

$$\begin{aligned} \tilde{A}_1^1 R + \tilde{B}_1 R P &= 0 \\ \tilde{A}_1^2 R &= 0 \end{aligned}$$

Hence $R \in \ker \tilde{A}_1^2$, and there exists a T such that $R = MT$, where $M = \mathrm{null}\, \tilde{A}_1^2$, and

$$\tilde{A}_1^1 M T + \tilde{B}_1 M T P = 0 \tag{22}$$

Theorem 6.1. *If the matrix pair $(A_1, A_2 + A_3)$ is regular, then matrices $\tilde{A}_1^1 M$ and $\tilde{B}_1 M$ are square and the matrix pair $(\tilde{A}_1^1 M, \tilde{B}_1 M)$ is regular.*

Proof. Let $\Gamma = [\mathrm{null}\, \tilde{A}_1^2, \mathrm{range}\, \tilde{A}_1^2]$. Matrix Γ is orthogonal. Let us consider matrices $\tilde{A}\Gamma$ and $\tilde{B}\Gamma$. We have

$$\tilde{A}\Gamma = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \qquad \tilde{B}\Gamma = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & 0 \end{bmatrix}$$

where $\tilde{A}_{11}$ and $\tilde{B}_{11}$ are square matrices with $n_1 = \tilde{A}_{11}$ rows. Let $\alpha, \beta \in C$. We have $|\det(\alpha A_1 - \beta(A_2 + A_3))| = |\det(\alpha\tilde{A} - \beta\tilde{B})| = |\det(\alpha\tilde{A}\Gamma - \beta\tilde{B}\Gamma)| = |\det(\alpha\tilde{A}_{11} - \beta\tilde{B}_{11}||\det(\alpha\tilde{A}_{22})|$.

Assume that $\mathrm{rank}\, \tilde{A}_1^2 < n - n_1$. Then $\mathrm{rank}\, M = m > n_1$. Then, by the construction of the matrix M, the first $m - n1 > 0$ columns of the matrix $\tilde{A}_{22}$ contain only zero elements. Hence, $|\det(\tilde{A}_{22})| = 0$, and the matrix pair $(A_1, A_2 + A_3)$ is not regular. On the other hand $\mathrm{rank}\, \tilde{A}_1^2 \leq n - n_1$, because the matrix $\tilde{A}_1^2$ contains $n - n_1$ rows. Thus $\mathrm{rank}\, \tilde{A}_1^2 = n - n_1$ and $\mathrm{rank}\, M = n_1$. The $n \times k$ matrix M has full column rank, thus $k = n_1$ and matrices $\tilde{A}_1^1 M$, $\tilde{B}_1 M$ are square.

If the matrix pair $(A_1, A_2 + A_3)$ is regular then there exist $\alpha, \beta \in C$ such that $|\det(\alpha A_1 - \beta(A_2 + A_3))| \neq 0$. Then also $|\det(\alpha\tilde{A}_{11} - \beta\tilde{B}_{11}| \neq 0$ and the matrix pair $(\tilde{A}_1^1 M, \tilde{B}_1 M)$ is regular. □

In this way we have proved that instead solving the problem (5) we can solve the problem (22) with respect to T and P. Then $R = MT$. If $\ker \tilde{B}_1 M \neq 0$, then we can repeat this procedure.

7 Models dependent on parameters

Let us assume that matrices describing the model (1) depend on additional scalar parameter, θ. Then equation 5 takes the form

$$0 = A_1(\theta)R(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)R(\theta)P(\theta) \tag{23}$$

equations (6-7) take the form

$$
\begin{aligned}
0 &= V_1(\theta) + A_1(\theta)S_1(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)R(\theta)Q_1(\theta) \\
0 &= A_1(\theta)S_3(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)R(\theta)Q_3(\theta) \\
0 &= V_2(\theta) + A_2(\theta)S_1(\theta) + A_2(\theta)R(\theta)Q_2(\theta) \\
0 &= A_2(\theta)S_3(\theta) + A_2(\theta)R(\theta)Q_4(\theta)
\end{aligned}
\tag{24}
$$

equations (8) take the form

$$
\begin{aligned}
0 &= A_1(\theta)G_1^0(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)R(\theta)F_1^0(\theta) + B_1(\theta) \\
0 &= A_2(\theta)R(\theta)F_2^0(\theta) + A_2(\theta)G_1^0(\theta) + B_2(\theta) \\
0 &= A_1(\theta)G_1^1(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)\Big(R(\theta)F_1^1(\theta) + G_1^0(\theta)\Big) + \Big(B_2(\theta) + B_3(\theta)\Big) \\
0 &= A_1(\theta)G_1^{k+2}(\theta) + \Big(A_2(\theta) + A_3(\theta)\Big)\Big(R(\theta)F_1^{k+2}(\theta) + G_1^{k+1}(\theta)\Big) \\
0 &= A_2(\theta)R(\theta)F_2^{k+1}(\theta) + A_2(\theta)G_1^{k+1}(\theta)
\end{aligned}
\tag{25}
$$

and the transversality condition takes the form

$$\lim_{t\to\infty} \xi(\theta)^t R(\theta)P(\theta)^t = 0$$

Let for the basic model $\theta = 0$. Let us assume that model matrices are differentiable with respect to θ, equation (5) has solutions R and P in some neighborhood of $\theta = 0$ and these solutions are differentiable with respect to θ[3]. Then equation (29) also has solution. We are going to expand matrices R, P, S_1, S_3, Q_1, Q_2, Q_3, Q_4, G_1^k, F_1^k, F_2^k in asymptotic series around $\theta = 0$. We already have the zero order terms.

Let $\bar{R} = R(0)$, $\bar{P} = P(0)$, $\bar{S}_1 = S_1(0)$, $\bar{S}_3 = S_3(0)$, $\bar{Q}_1 = Q_1(0)$, $\bar{Q}_2 = Q_2(0)$, $\bar{Q}_3 = Q_3(0)$, $\bar{Q}_4 = Q_4(0)$, $\bar{G}_1^k = G_1^k(0)$, $\bar{F}_1^k = F_1^k(0)$, $\bar{F}_2^k = F_2^k(0)$ solve (23-25). Differentiating (23-24) with respect to θ around $\theta = 0$ yields

$$
\begin{aligned}
0 = {} & A_1'(0)\bar{R} + (A_2'(0) + A_3'(0))\bar{R}\bar{P} + A_1R'(0) + (A_2 + A_3)R'(0)\bar{P} \\
& + (B + C)\bar{R}P'(0)
\end{aligned}
\tag{26}
$$

[3]Generally this conditions need not be fulfilled. E.g. a model $0 = y_t + \theta\epsilon_{t+1}$ has solution only for $\theta = 0$, a model $\theta = E_t y_{t+1}$ may have solution $y_t = \theta + \sigma(\theta)\epsilon_t$, where the function $\sigma(\theta)$ is not differentiable.

and

$$
\begin{aligned}
0 &= V_1'(0) + A_1'(0)\bar{S}_1 + \Big(A_2'(0) + A_3'(0)\Big)\bar{R}\bar{Q}_1 + (A_2 + A_3)R'(0)\bar{Q}_1 \\
&\quad + A_1 S_1'(0) + (A_2 + A_3)\bar{R}Q_1'(0) \\
0 &= A_1'(0)\bar{S}_3 + \Big(A_2'(0) + A_3'(0)\Big)\bar{R}\bar{Q}_3 + (A_2 + A_3)R'(0)\bar{Q}_3 \\
&\quad + A_1 S_3'(0) + (A_2 + A_3)RQ_3'(0) \\
0 &= V_2'(0) + A_2'(0)\bar{S}_1 + A_2 R'(0)\bar{Q}_2 + A_2 S_1'(0) + A_2\bar{R}Q_2'(0) \\
0 &= A_2'\bar{S}_3 + A_2 R'(0)\bar{Q}_4 + A_2 S_3'(0) + A_2\bar{R}Q_4'(0)
\end{aligned}
\tag{27}
$$

where $A_1 = A(0)$, $A_2 = A_2(0)$, $A_3 = A_3(0)$, $V_1 = V_1(0)$, $V_2 = V_2(0)$. Equations (27) have the same structure as equations (6-7) and can be solved using methods from the section 4[4].

Differentiating (25) with respect to θ around $\theta = 0$ yields

$$
\begin{aligned}
0 &= A_1'(0)\bar{G}_1^0 + \Big(A_2'(0) + A_3'(0)\Big)\bar{R}\bar{F}_1^0 + (A_2 + A_3)R'(0)\bar{F}_1^0 + B_1'(0) \\
&\quad + A_1(G_1^0)'(0) + (A_2 + A_3)\bar{R}(F_1^0)'(0) \\
0 &= A_2'(0)\bar{R}\bar{F}_2^0 + A_2 R'(0)\bar{F}_2^0 + A_2'(0)\bar{G}_1^0 + B_2'(0) + A_2\bar{R}(F_2^0)'(0) + A_2(G_1^0)'(0) \\
0 &= A_1'(0)\bar{G}_1^1 + \Big(A_2'(0) + A_3'(0)\Big)(\bar{R}\bar{F}_1^1 + \bar{G}_1^0) + (A_2 + A_3)R'(0)\bar{F}_1^1 + \Big(B_2'(0) + B_3'(0)\Big) \\
&\quad + (A_2 + A_3)(G_1^0)'(0) + A_1(G_1^1)'(0) + (A_2 + A_3)\bar{R}(F_1^1)'(0) \\
0 &= A_1'(0)\bar{G}_1^{k+2} + \Big(A_2'(0) + A_3'(0)\Big)\Big(\bar{R}\bar{F}_1^{k+2} + \bar{G}_1^{k+1}\Big) + (A_2 + A_3)R'(0)\bar{F}_1^{k+2} \\
&\quad + (A_2 + A_3)(G_1^{k+1})'(0) + A_1(G_1^{k+2})'(0) + (A_2 + A_3)\bar{R}(F_1^{k+2})'(0) \\
0 &= A_2'(0)\bar{R}\bar{F}_2^{k+1} + A_2 R'(0)\bar{F}_2^{k+1} + A_2'(0)\bar{G}_1^{k+1} + A_2\bar{R}(F_2^{k+1})'(0) + A_2(G_1^{k+1})'(0)
\end{aligned}
\tag{28}
$$

Equations (28) have the same structure as equations (8) and can be solved using methods from the section 5.

Equation (26) takes the form

$$
0 = \Gamma^1 + A_1 X + (A_2 + A_3)X\bar{P} + (A_2 + A_3)\bar{R}Y \tag{29}
$$

where $\Gamma^1 = A_1'(0)\bar{R} + (A_2'(0) + A_3'(0))\bar{R}\bar{P}$, $X = R'(0)$, and $Y = P'(0)$, with unknown matrices X and Y. There are many solutions to (29). Observe that if X is a solution to (29), then $X + \alpha\bar{R}$ is also a solution for any α. To avoid this indeterminacy we can utilize the fact, that if $R(\theta)$ is a solution to (5), then

[4]Generally there may be infinitely many solutions to (27), for example in case of a regular model $\theta = E_t y_{t+1}$, with one of the solution $y_t = \theta + \alpha\theta\epsilon_t$ for any α.

$R(\theta)U$ is also a solution for any invertible matrix U. Let $M(\theta)$ spans the range of $R(\theta)'$. Then $M(\theta)'R(\theta)$ is an invertible matrix. Let us assume that in some neighborhood of $\theta = 0$ the matrix $M(0)'R(\theta)$ is invertible. Then there exists an invertible matrix $U(\theta)$ such that $M(0)'R(\theta)U(\theta) = I$ and as a new solution we can take $\tilde{R}(\theta) = R(\theta)U(\theta)U^{-1}(0)$. Then $\tilde{R}(0) = \bar{R}$ and $M(0)'\tilde{R}'(0) = 0$. Hence, we can assume that $M(0)'X = 0$. Thus,

$$X = K\tilde{X}$$

where K spans the null space of $\bar{R}'$. Equation (29) now takes the form

$$0 = \Gamma^1 + A_1K\tilde{X} + (A_2 + A_3)K\tilde{X}\bar{P} + (A_2 + A_3)\bar{R}Y$$

We are going to express the matrix Y in terms of $\tilde{X}$. Let matrices N, M span the null space and the range of $(A_2 + A_3)'$. Then

$$0 = N'\Gamma^1 + N'A_1K\tilde{X} \tag{30}$$

$$0 = M'\Gamma^1 + M'A_1K\tilde{X} + M'(A_2 + A_3)K\tilde{X}\bar{P} + M'(A_2 + A_3)\bar{R}Y \tag{31}$$

Proposition 7.1. *If the matrix pair* $(A_1, A_2 + A_3)$ *is regular, then* $\ker(N'A_1K)' = 0$.

Proof. Let $\Gamma = \mathrm{col}(N', M')$. Then Γ is square orthogonal matrix and

$$\Gamma A_1 = \begin{bmatrix} N'A_1 \\ \tilde{A}_2 \end{bmatrix}, \qquad \Gamma(A_2 + A_3) = \begin{bmatrix} 0 \\ \tilde{B}_2 \end{bmatrix}$$

Let $\ker(N'A_1)' \neq 0$. Let $\bar{N}$, $\bar{M}$ span null space and range of $(N'A)'$ and let

$$\Delta = \begin{bmatrix} \bar{N}' & 0 \\ \bar{M}' & 0 \\ 0 & I \end{bmatrix}$$

Then Δ is a square orthogonal matrix and

$$\Delta\Gamma A_1 = \begin{bmatrix} 0 \\ \bar{A}_2 \end{bmatrix}, \qquad \Delta\Gamma(A_2 + A_3) = \begin{bmatrix} 0 \\ \tilde{B}_2 \end{bmatrix}$$

Let α, β are any complex scalars. Then $\det(\alpha A_1 - \beta(A_2 + A_3)) = \det(\alpha\Delta\Gamma A_1 - \beta\Delta\Gamma(A_2 + A_3)) = 0$ and the matrix pair $(A_1, A_2 + A_3)$ is not regular. Thus, $\ker(N'A_1)' = 0$.

Let $x \in \ker(N'A_1K)'$. Then $0 = K'(N'A_1)'x$ and $x \in \ker(N'A_1)'$, because $\ker K = 0$ by construction. Hence $x = 0$. □

Proposition 7.1 implies that the equation (30) always has at least one solution. Let

$$\tilde{X} = G + HZ$$

for appropriate matrices G, H and any matrix Z.

Now the equation (31) takes the form

$$0 = \Gamma_2 + M'A_1KHZ + M'(A_2 + A_3)KHZ\bar{P} + M'(A_2 + A_3)\bar{R}Y \tag{32}$$

where $\Gamma_2 = M'\Gamma^1 + M'A_1KG + M'(A_2+A_3)KG\bar{P}$ with unknown Y and Z. Again let matrices $\tilde{N}$, $\tilde{M}$ span the null space and the range of $(M'(A_2 + A_3)\bar{R})'$. Then

$$0 = \tilde{N}'\Gamma_2 + \tilde{N}'M'A_1KHZ + \tilde{N}'M'(A_2 + A_3)KHZ\bar{P} \tag{33}$$

and

$$0 = \tilde{M}'\Gamma_2 + \tilde{M}'M'A_1KHZ + \tilde{M}'M'(A_2 + A_3)KHZ\bar{P} + \tilde{M}'M'(A_2 + A_3)\bar{R}Y \tag{34}$$

Proposition 7.2. *If the matrix pair $(A_1, A_2 + A_3)$ is regular, then for any Z the equation (34) has exactly one solution.*

Proof. By the construction the matrix $\tilde{M}'M'(A_2 + A_3)\bar{R}$ has full row rank, and the equation (34) has at least one solution for any Z. Assume that there exist a second solution $\tilde{Y} \neq Y$. Then $\tilde{M}'M'(A_2+A_3)\bar{R}(Y-\tilde{Y}) = 0$. The matrix $\tilde{M}'$ spans range of $M'(A_2 + A_3)\bar{R}$, hence $M'(A_2 + A_3)\bar{R}(Y - \tilde{Y}) = 0$. The matrix M' spans range of $A_2 + A_3$, hence $(A_2 + A_3)\bar{R}(Y - \tilde{Y}) = 0$. This contradicts the proposition 3.3. □

By the proposition 7.2 the matrix $\tilde{M}'M'(A_2 + A_3)\bar{R}$ is square and invertible.

Proposition 7.3. *If the matrix pair (A_1, A_2+A_3) is regular, then matrices $\tilde{N}'M'A_1KH$ and $\tilde{N}'M'(A_2 + A_3)KH$ are square.*

Proof. We need to prove that $\dim_1 \tilde{N}' = \dim_2 H$. This is equivalent to $\dim_2 \operatorname{null}(\bar{R}'(A_1+A_2)'M) = \dim_2 \operatorname{null} N'A_1K$. By the proposition 7.2 the matrix $\tilde{M}'M'(A_2 + A_3)\bar{R}$ is square and invertible, hence the matrix $Q = \bar{R}'(A_2 + A_3)'M$ has full row rank and $\dim_2 \operatorname{null}(\bar{R}'(A_2 + A_3)'M) = \dim_2 Q - \dim_1 Q = \dim_2 M - \dim_2 \bar{R}$.

To prove that also $\operatorname{null} N'A_1K$ has full rank let us consider the Schur decomposition of the matrix pair $(A_1, A_2 + A_3)$.

$$\begin{aligned} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_A & T_{12}^A \\ 0 & T_{22}^A \end{bmatrix} &= A_1 \begin{bmatrix} U_1 & U_2 \end{bmatrix} \\ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_B & T_{12}^B \\ 0 & T_{22}^B \end{bmatrix} &= (A_2 + A_3) \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{aligned} \tag{35}$$

where matrices $U = [U_1, U_2]$, $V = [V_1, V_2]$ are orthogonal, R_A is quasi-upper triangular, R_B is upper-triangular and invertable, both matrices have the same size and eigenvalues are such selected, that the growth restriction holds. Then $R = U_1\Xi$ for some invertible matrix Ξ (see section 3). In this case $K = \operatorname{null} R' = U_2$. Hence

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} T_{12}^A \\ T_{22}^A \end{bmatrix} = A_1 K$$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} T_{12}^B \\ T_{22}^B \end{bmatrix} = (A_2 + A_3) K$$

The matrix N spans the null space of $(A_2 + A_3)'$. Because the matrix R_B is invertible, thus $N = V_2\Lambda$ for some orthogonal matrix Λ. In this way we have

$$T_{22}^A = V_2' A_1 K \qquad\qquad T_{22}^B = V_2'(A_2 + A_3)K$$

and further

$$\Lambda' T_{22}^A = N' A_1 K \qquad\qquad \Lambda' T_{22}^B = 0$$

Regularity of the matrix pair $(A_1, A_2 + A_3)$ requires that $\Lambda' T_{22}^A$ has full row rank. This ends proof that $S = N'AK$ has full row rank. Thus $\dim_2 \operatorname{null}(N'AK) = \dim_2 S - \dim_1 S = \dim_2 K - \dim_2 N$.

Finally we have following relations: $\dim_2 N + \dim_2 M = \dim_1 A_2$ and $\dim_2 \bar{R} + \dim_2 K = \dim_1 A_2$. Hence $\dim_2 M - \dim_2 \bar{R} = \dim_2 K - \dim_2 N$ and $\dim_1 \tilde{N}' = \dim_2 H$. □

Let us solve now the equation (33). We can solve this equation using vectorization technique obtaining

$$0 = \operatorname{vec}(\tilde{N}'\Gamma_2) + (I \otimes \tilde{N}'M'AKH + (\bar{P})' \otimes \tilde{N}'M'(B + C)KH) \operatorname{vec}(Z)$$

This equation can be solved using methods from the section 4, which also show whether there is any solution.

However by the theorem 7.3 we know that equation (33) is the generalized Sylvester equation, which can be solved more efficiently especially in case of large dense problems. Generally the equation (33) may have zero, one, or infinitely many solutions.

For small θ the modified transversality implies choosing the same eigenvalues of the matrix pair $(A_1(\theta), A_2(\theta) + A_3(\theta))$ as in case $\theta = 0$.

Now we can expressed matrices describing solution to the model (5) as

$$\begin{aligned} R(\theta) &\sim \bar{R} + \theta R'(0) & P(\theta) &\sim \bar{P} + \theta P'(0) \\ S_i(\theta) &\sim \bar{S}_i + \theta S_i'(0) & Q_j(\theta) &\sim \bar{Q}_j + \theta Q_j'(0) \\ G_1^k(\theta) &\sim \bar{G}_1^k + \theta (G_1^k)'(0) & F_1^k(\theta) &\sim \bar{F}_1^k + \theta (F_1^k)'(0) \\ F_2^k(\theta) &\sim \bar{F}_2^k + \theta (F_2^k)'(0) \end{aligned}$$

for $\theta \to 0$ and $i = 1, 2$. Such a result is very useful in estimating linear model. When there are many parameters in the model then we can repeat the procedure for all parameters.

8 Predefined state variables

Usually we would like to represent model dynamics in terms of predefined state variables. Assume that we interpret some endogenous variables as a state variables, x_t

$$x_t = K y_t$$

which in time $t = 0$ may take any values. In generality model dynamics does not depends only on these state variables. They may appear additional state variables representing sunspots. On the other hand we would like not to introduce predefined state variables too early, because in some cases it is difficult to identify variables, which values are predetermined in given period and can take any value in period $t = 0$.

We have

$$x_t = K R u_t$$

By the assumption, all values x_t are possible, hence KR must have full row rank. Let V is an orthogonal matrix such that $KR = TV'$, where $T = [\tilde{T}, 0]$ and $\tilde{T}$ is an invertible matrix. Since KR has full row rank, such a matrix V exists. Then

$$T^{-1} x_t = V_1' u_t$$

where $V = \mathrm{col}(V_1, V_2)$ is partition of the matrix corresponding to partition of the matrix T. Let $u_t = V \tilde{u}_t$. Then

$$\tilde{T}^{-1} x_t = V_1' V \tilde{u}_t = \tilde{u}_t^1$$

where $\tilde{u}_t = [\tilde{u}_t^1, \tilde{u}_t^2]'$ is partition of the vector $\tilde{u}_t$ corresponding to partition of matrices T and V. On the other hand we have

$$\tilde{u}_t = V' P V \tilde{u}_{t-1} + V' Q \upsilon_t \equiv \tilde{P} \tilde{u}_{t-1} + \tilde{Q} \upsilon_t$$

where $Q\upsilon_t$ represents stochastic and exogenous part of state variables dynamics. Hence

$$\begin{aligned} x_t &= \tilde{T} \tilde{P}_{11} \tilde{T}^{-1} x_{t-1} + \tilde{T} \tilde{P}_{12} \tilde{u}_{t-1}^2 + \tilde{T} \tilde{Q}^1 \upsilon_t \\ \tilde{u}_t^2 &= \tilde{P}_{21} \tilde{T}^{-1} x_{t-1} + \tilde{P}_{22} \tilde{u}_{t-1}^2 + \tilde{Q}^2 \upsilon_t \end{aligned}$$

where $\tilde{Q} = \mathrm{col}(\tilde{Q}^1, \tilde{Q}^2)$ is partition of the matrix $\tilde{Q}$ corresponding to partition of the matrix T and

$$\begin{bmatrix} x_t \\ \tilde{u}_t^2 \end{bmatrix} = \begin{bmatrix} \tilde{T}\tilde{P}_{11}\tilde{T}^{-1} & \tilde{T}\tilde{P}_{12} \\ \tilde{P}_{21}\tilde{T}^{-1} & \tilde{P}_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \tilde{u}_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{T}\tilde{Q}^1 \\ \tilde{Q}^2 \end{bmatrix} \upsilon_t$$

We have also

$$\begin{aligned} y_t &= Ru_t + S\upsilon_t = RV \begin{bmatrix} \tilde{u}_t^1 \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t = R \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{T}^{-1}x_t \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t \\ &= R \begin{bmatrix} V_1\tilde{T}^{-1} & V_2 \end{bmatrix} \begin{bmatrix} x_t \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t \end{aligned}$$

where $S\upsilon_t$ represents stochastic and exogenous part of endogenous variables dynamics.

We do not calculate expansion of matrices describing solution to the model depending on additional parameters with predefined variables as state variables, because such a representation of model dynamics is not useful in estimation of model parameters.

9 Conclusions

We presented algorithm of analyzing general set of linear dynamic rational expectations models. We concentrated only on regular models. This excludes models with larger set of indeterminacy, e.g. models with many capital assets. Such models are however very pathological since they do not uniquely determine solution to deterministic part. Solving such models would require analyzing the Kronecker decomposition of a matrix pair using ordered GUPTRI decomposition, which to our knowledge is not available yet.

References

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